COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **22** (1) (2002) 35–46

HOMOMORPHISMS OF PRODUCTS OF GRAPHS INTO GRAPHS WITHOUT FOUR CYCLES

CHRISTIAN DELHOMMÉ, NORBERT SAUER[†]

Received August 31, 1998
Revised April 19, 2000

Given two graphs A and G, we write $A \mapsto G$ if there is a homomorphism of A to G and $A \not\mapsto G$ if there is no such homomorphism. The graph G is C_4-free if, whenever both a and c are adjacent to b and d, then $a=c$ or $b=d$. We will prove that if A and B are connected graphs, each containing a triangle and if G is a C_4-free graph with $A \not\mapsto G$ and $B \not\mapsto G$, then $A \times B \not\mapsto G$ (here "$\times$" denotes the categorical product).

Introduction

A *graph* G is a pair consisting of the set of *vertices* $V(G)$ and the set of *edges* $E(G) \subseteq [V(G)]^2$. Given two vertices x and y, we write $x \sim y$ if $\{x,y\} \in E(G)$. The graph H is a *subgraph* of the graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The graph G is *C_4-free* if $a \sim b \sim c \sim d \sim a$ implies $a=c$ or $b=d$. The function $f : V(A) \mapsto V(G)$ is a *homomorphism* of the graph A to the graph G if $a \sim b$ implies $f(a) \sim f(b)$. We write $A \mapsto G$ if there is a homomorphism of the graph A to the graph G and $A \not\mapsto G$ if there is no such homomorphism. Given two graphs A and B we denote by $A \times B$ the graph with $V(A \times B) = V(A) \times V(B)$ and $E(A \times B) = \{\{(a_0,b_0),(a_1,b_1)\} \mid \{a_0,a_1\} \in E(A) \text{ and } \{b_0,b_1\} \in E(B)\}$. (We will use the symbol "$\times$" for both the cartesian product of sets and the product of graphs.)

The chromatic number of the graph $A \times B$ is easily seen to be less than or equal to the minimum of the chromatic numbers of the graphs A and B.

Mathematics Subject Classification (2000): 05C15

[†] Supported by NSERC of Canada Grant #691325.

The long lasting conjecture of Hedetniemi says that *the chromatic number of the graph $A\times B$ is equal to the minimum of the chromatic numbers of the graphs A and B.* With K_n the complete graph on n vertices, Hedetniemi's conjecture can be stated as $A\times B\mapsto K_n$ *implies* $A\mapsto K_n$ *or* $B\mapsto K_n$. A graph G is *multiplicative* if $A\times B\mapsto G$ implies $A\mapsto G$ or $B\mapsto G$. Multiplicative graphs are of interest as they are the meet irreducible graphs in the lattice of graphs with the preordering relation determined by homomorphisms between graphs. (See [1] or [2] for this and a more complete list of references.)

The graph G is *multiplicative in the class* $\mathbf{S}$ *of graphs* if ($A\in\mathbf{S}$, $B\in\mathbf{S}$ and $A\times B\mapsto G$) implies ($A\mapsto G$ or $B\mapsto G$). In particular (by compactness), a finite graph is multiplicative whenever it is multiplicative in the class of all finite graphs. It is not known whether K_n is multiplicative when $n\geq 4$. It is easy to see that K_1 and K_2 are multiplicative and it is proven in [3] that K_3 is multiplicative. Actually, there are no multiplicative graphs known which have a chromatic number larger than three. We do not produce graphs which are multiplicative and have large chromatic number, but still we obtain, as a consequence of the main theorem,

Theorem. *Every C_4-free graph is multiplicative in the class of graphs which contain a triangle in every connected component.*

The proof of the Theorem requires notation and methods not quite common in graph theory. What follows is motivation for the notation and an outline of the proof.

We wish to find conditions assuring that, if f is a homomorphism of the product of the graphs A and B to the graph G, then either $A\mapsto G$ or $B\mapsto G$. If we fix a vertex $b\in V(B)$ we obtain a function f_b of $V(A)$ to $V(G)$ given by $f_b(a)=f(a,b)$.

The result of [3] can be stated as: *Given two graphs A and B, if $A\times B\mapsto K_3$ then either $A\mapsto K_3$ or $B\mapsto K_3$.* In order to prove this, a "parity", odd or even, is defined for every function f, not necessarily a homomorphism, of a circuit to K_3. It turns out that if f is a homomorphism of the product of a circuit with the path $P_{a,b}$ (on two vertices a and b), then the parity of f_a is equal to the parity of f_b. This implies that, if f is a homomorphism of the product $[c_0,c_1,\dots,c_{n-1},c_0]\times[d_0,d_1,\dots,d_{m-1},d_0]$ of two circuits, then the parity of f_{c_i} is equal to the parity of f_{c_j} for $i,j\in n$ and the parity of f_{d_i} is equal to the parity of f_{d_j} for $i,j\in m$. Hence we can speak of the parity of the circuit $[c_0,c_1,\dots,c_{n-1},c_0]$ and of the parity of the circuit $[d_0,d_1,\dots,d_{m-1},d_0]$ under the homomorphism f. Furthermore if n and m are odd, then the two parities must be different (similar to Lemma 5 in this paper). Hence, when A and B are connected, if f is a homomorphism of $A\times B$ to K_3, then the parities under f of all odd circuits in one of the two graphs, say A, must be odd. The proof is then finished by showing that this implies that $A \mapsto K_3$. We follow this last part via Lemma 6 and Lemma 7.

Our proof here generalizes the ideas of the proof in [3]. We set out to find an invariant similar to the parity in [3]. A closer examination of the notion of parity of a function g of a circuit, say of odd length ℓ, to K_3 shows that it is the parity of a winding number, a standard concept in topology : as we move around the circuit with length two steps, our image under g proceeds along the triangle; the parity of g is then the parity of the number of times that our image has effectively wound around the triangle until we reach our starting point, that is, after ℓ steps. In case g takes values in the vertex set of another odd circuit, say C, and provided that each length two step that we take corresponds to a length zero or two step in C for g (which will indeed be the case in our context, since g will be f_b for some non isolated b), the parity to consider corresponds to the rotation around the circuit obtained from C by linking two points at distance two (see [4]).

Note that if C is an odd circuit then $C \times P_{a,b}$ is an even circuit twice the length of C. Let f be a homomorphism of $C \times P_{a,b}$ to a graph G. We endow C with a circular orientation which then induces a circular orientation around $C \times P_{a,b}$. Then we trace the closed directed walk $f(V(\overrightarrow{C} \times P_{a,b}))$ as we go around $\overrightarrow{C} \times P_{a,b}$ in the direction of the chosen orientation. As we trace this directed walk in G, we formally add up the edges of G that we encounter; in particular, if we go forward along the edge say (x,y) and then later on backward along the same edge, that is along (y,x), then we count that as 0, that is no progress along (x,y) or (y,x); in other words, the edge (y,x) is considered as the opposite of the edge (x,y) and addition of edges is commutative: we work in the Abelian group generated by the edges of G subject to the relations "$(x,y)+(y,x)=0$". Since $C \times P_{a,b}$ is an even circuit, we are left with a sum of edges containing an even number of terms. Half this number will be our parity. This parity can be informally considered as the parity of half "the number of times the walk $f(V(\overrightarrow{C} \times P_{a,b}))$ winds around odd circuits of G". Now, when f prolongs to a homomorphism from a product $C \times D$, where D is an odd circuit containing the path $P_{a,b}$, and when further G is C_4-free, the walk has in fact "wound an even number of times around odd circuits of G". In order to see this we shall impose some restriction on cancellation between "opposite edges". Let us look at the situation more closely. Let f be a homomorphism of the product of the circuit $[c_0, c_1, \ldots, c_{n-1}, c_0]$ with the path $P_{a,b}$ to G. There are two kinds of edges in this product: edges either "go from a to b" (these are the edges of the form $((c_i,a),(c_{i+1},b))$), or "go from b to a" (the edges of the form $((c_i,b),(c_{i+1},a))$). We shall allow cancellations only between the images of two edges of different types. With these restrictions, we obtain the following: Let $P_{a,b,c}$ be a path, let C be a circuit and let f be a homomorphism of $C \times P_{a,b,c}$ to a C_4-free G; then the sum of edges of G, obtained by applying f to $C \times P_{a,b}$ is the same as the sum obtained when f is applied to $C \times P_{c,b}$. This provides the desired invariant and then, the argument follows the one in [3].

Notation

A *directed graph*, $\overrightarrow{G}$, is a pair consisting of the set of *vertices* $V(\overrightarrow{G})$ and the set of *directed edges* or *edges* $E(\overrightarrow{G}) \subseteq V(\overrightarrow{G}) \times V(\overrightarrow{G})$. The directed graph $\overrightarrow{G}$ is an *oriented graph* if $(x,y) \in E(\overrightarrow{G})$ implies $(y,x) \notin E(\overrightarrow{G})$. Let G be the graph with $V(G) = V(\overrightarrow{G})$ and $E(G) = \{\{x,y\} \mid (x,y) \in E(\overrightarrow{G})\}$. We say that the graph G is obtained from the oriented graph $\overrightarrow{G}$ by *forgetting* the orientation of the edges. Of course different orientations of the edges of G give rise to different oriented graphs.

$P_{a,b}$ will denote the (undirected) graph with vertex set $\{a,b\}$ (assuming $a \neq b$) and with edge $\{a,b\}$. $P_{a,b,c}$ will denote the graph with vertex set $\{a,b,c\}$ and edge set $\{\{a,b\},\{b,c\}\}$ (we assume that $a \neq b$ and $b \neq c$, but a and c may be equal). We denote by $\langle c_0, c_1, \ldots, c_{n-1}, c_0 \rangle$ the *cycle* with vertex set $\{c_i \mid i \in n\}$ and edge set $\{(c_i, c_{i+1}) \mid i \in n$ (addition modulo (n))$\}$. (We assume that the c_i's are pairwise distinct. A cycle is an oriented graph.) A cycle with n vertices will be denoted by $\overrightarrow{C}_n$. We say $\overrightarrow{C}_n$ is a cycle of a graph A if C_n is a subgraph of A. A *walk* (of length m) in a graph A is any sequence $(a_0, a_1, \ldots, a_m)$ of vertices of A such that $a_i \sim a_{i+1}$, for every $i \in m$.

Let $\overrightarrow{A}$ be a directed graph and B a graph. We say that f is a homomorphism of $\overrightarrow{A}$ to B if f is a homomorphism of A to B. The oriented graph $\overrightarrow{A} \times B$ has vertex set $V(\overrightarrow{A} \times B) = V(\overrightarrow{A}) \times V(B)$ and edge set $E(\overrightarrow{A} \times B) = \Big\{((a_0,b_0),(a_1,b_1)) \mid (a_0,a_1) \in E(\overrightarrow{A}) \text{ and } \{b_0,b_1\} \in E(B)\Big\}$. Thus the graph $A \times B$ is obtained from the graph $\overrightarrow{A} \times B$ by forgetting the orientation of the edges, and the orientation of the edges in $\overrightarrow{A} \times B$ is determined by the orientation of the edges in $\overrightarrow{A}$.

For each integer $n \in \mathbf{Z}$ we denote by $\overline{n} \in \mathbf{Z}/2\mathbf{Z}$ its class modulo 2. Given a set X, denote by $S(X)$ the Abelian group generated by the set $X \times X$ of ordered pairs of elements of X, subject to the relations $(x,y)+(y,x)=0$ and $(x,x)=0$; $S(X)$ is indeed isomorphic to the Abelian group freely generated by the set of unordered pairs of elements of X. This group is endowed with an integer valued norm, which, to each *chain* $\mathbf{v} \in S(X)$, associates $\|\mathbf{v}\|$, the least non-negative integer d such that $\mathbf{v}$ is equal to the sum $(x_0,y_0)+(x_1,y_1)+$

$\ldots + (x_{d-1}, y_{d-1})$. If a chain $\mathbf{v}$ is written as a sum of $\|\mathbf{v}\|$ edges then $\mathbf{v}$ is in *normal form*. Any two normal forms of $\mathbf{v}$ differ only by the order of the terms in the sum. Observe that $\|\mathbf{v}+(x,y)\| = \|\mathbf{v}\| \pm 1$, for each ordered pair $(x,y) \in X \times X$ and each chain $\mathbf{v} \in S(X)$. It follows that $\mathbf{v} \mapsto \overline{\|\mathbf{v}\|}$ is a group homomorphism of $S(X)$ to $\mathbf{Z}/2\mathbf{Z}$.

Let $\pi : X \times \{0,1\} \to X$and $\sigma : X \times \{0,1\} \to X \times \{0,1\}$, be given by $\pi(x,i) = x$ and $\sigma(x,i) = (x, 1-i)$. Every mapping f of some set X to some set Y gives rise to a group homomorphism of $S(X)$ to $S(Y)$ by stipulating that $f(a,b) = (f(a), f(b))$ for every $(a,b) \in X \times X$ (and $f(\mathbf{v}+\mathbf{w}) = f(\mathbf{v}) + f(\mathbf{w})$ for any two chains $\mathbf{v}$ and $\mathbf{w}$ in $S(X)$).

For every finite oriented graph $\overrightarrow{A}$, the chain of $S\left(V(\overrightarrow{A})\right)$ equal to the sum of all (oriented) edges of $\overrightarrow{A}$ will be denoted by $\sum \overrightarrow{A}$.

Lemma 1. *Let f be a function of the set X to the set Y and $\mathbf{v}$ a chain of $S(X)$ such that $f(x) \neq f(y)$ for every (x,y) occurring in the normal form of $\mathbf{v}$. Then $\overline{\|\mathbf{v}\|} = \overline{\|f(\mathbf{v})\|}$.*

Proof. Induction on $\|\mathbf{v}\|$. Observe that

$$\begin{aligned} \overline{\|f(\mathbf{v}+(x,y))\|} &= \overline{\|f(\mathbf{v}) + (f(x), f(y))\|} \\ &= \overline{\|f(\mathbf{v})\|} + \overline{\|(f(x), f(y))\|} = \overline{\|f(\mathbf{v})\|} + \overline{1} \end{aligned}$$

whereas $\overline{\|\mathbf{v}+(x,y)\|} = \overline{\|\mathbf{v}\|} + \overline{\|(x,y)\|} = \overline{\|\mathbf{v}\|} + \overline{1}$. ∎

Let $\overrightarrow{C_n} = \langle c_0, c_1, \ldots, c_{n-1}, c_0 \rangle$ be a cycle. Every homomorphism f of $\overrightarrow{C}_n \times P_{a,b}$ to a graph G gives rise to a homomorphism $f_{(a,b)}$ of $\overrightarrow{C}_n \times P_{a,b}$ to $G \times P_{0,1}$ where

$$f_{(a,b)}(c_i, x) := \begin{cases} (f(c_i,x), 0), & \text{if } x = a, \\ (f(c_i,x), 1), & \text{if } x = b. \end{cases}$$

Notice that $\pi \circ f_{(a,b)} = f$, that $\sigma \circ f_{(a,b)} = f_{(b,a)}$ and hence that

$$f\left(\sum(\overrightarrow{C}_n \times P_{a,b})\right) = \pi\left(f_{(a,b)}\left(\sum(\overrightarrow{C}_n \times P_{a,b})\right)\right) \tag{1}$$

and that

$$\sigma\left(f_{(a,b)}\left(\sum(\overrightarrow{C}_n \times P_{a,b})\right)\right) = f_{(b,a)}\left(\sum(\overrightarrow{C}_n \times P_{a,b})\right). \tag{2}$$

Besides

$$\sum(\overrightarrow{C}_n \times P_{a,b}) = \sum_{i \in n} ((c_i, a), (c_{i+1}, b)) + \sum_{i \in n} ((c_i, b), (c_{i+1}, a))$$

with the subscript addition modulo n. Hence

$$f\left(\sum(\overrightarrow{C}_n \times P_{a,b})\right) \\ = \sum_{i\in n}(f((c_i,a)),f((c_{i+1},b))) + \sum_{i\in n}(f((c_i,b)),f((c_{i+1},a)))$$

and

$$f_{(a,b)}\left(\sum(\overrightarrow{C}_n \times P_{a,b})\right) = \\ = \sum_{i\in n}(f(((c_i,a),0)),f(((c_{i+1},b),1))) + \sum_{i\in n}(f(((c_i,b),1)),f(((c_{i+1},a),0)))$$

and

$$f_{(b,a)}\left(\sum(\overrightarrow{C}_n \times P_{a,b})\right) = \\ = \sum_{i\in n}(f(((c_i,a),1)),f(((c_{i+1},b),0)))+\sum_{i\in n}(f(((c_i,b),0)),f(((c_{i+1},a),1)))\,.$$

Proof of the Theorem

Lemma 2. *If f is a homomorphism of $\overrightarrow{C}_n \times P_{a,b,c}$ to some C_4-free graph G, then*

$$f_{(a,b)}\left(\sum(\overrightarrow{C}_n \times P_{a,b})\right) = f_{(c,b)}\left(\sum(\overrightarrow{C}_n \times P_{b,c})\right).$$

Proof. Assume that $\overrightarrow{C}_n = \langle 0,1,\dots,n-1,0\rangle$. Let, for each $i\in n$,

$$D_i := ((i,a),(i+1,b))+((i+1,b),(i,c))+((i,c),(i-1,b))+((i-1,b),(i,a))$$

and

$$\begin{aligned} f[D_i] \;:=\; & f_{(a,b)}\left((i,a),(i+1,b)\right) + f_{(c,b)}\left((i+1,b),(i,c)\right) + \\ & +f_{(c,b)}\left((i,c),(i-1,b)\right) + f_{(a,b)}\left(i-1,b),(i,a)\right) = \\ & ((f(i,a),0),(f(i+1,b),1)) + ((f(i+1,b),1),(f(i,c),0)) + \\ & + ((f(i,c),0),(f(i-1,b),1)) + ((f(i-1,b),1),(f(i,a),0))\,. \end{aligned} \tag{3}$$

Since the function f is a homomorphism of $\overrightarrow{C}_n \times P_{a,b,c}$ to G and $\{(i,a)\sim(i+1,b)\sim(i,c)\sim(i-1,b)\sim(i,a)\}$ in $C_n \times P_{a,b,c}$, it follows that $f(i,a)\sim f(i+1,b)\sim f(i,c)\sim f(i-1,b)\sim f(i,a)$ in the graph G. Because G is C_4-free $f(i,a)=f(i,c)$ or $f(i+1,b)=f(i-1,b)$ which implies, using the last two lines of (3), that $f[D_i]=0$ for all $i\in n$. Note that

$$\begin{aligned}\sum_{i\in n} f[D_i] &= \sum_{i\in n}\Big(f_{(a,b)}\left((i,a),(i+1,b)\right)+f_{(a,b)}\left((i-1,b),(i,a)\right)\Big)+\\ &\qquad +\sum_{i\in n}\Big(f_{(c,b)}\left((i+1,b),(i,c)\right)+f_{(c,b)}\left((i,c),(i-1,b)\right)\Big)=\\ &= f_{(a,b)}\sum_{i\in n}\left(((i,a),(i+1,b))+((i-1,b),(i,a))\right)+\\ &\qquad +f_{(c,b)}\sum_{i\in n}\left(((i+1,b),(i,c))+((i,c),(i-1,b))\right)=\\ &= f_{(a,b)}\left(\sum(\overrightarrow{C}_n\times P_{a,b})\right)+f_{(c,b)}\left(\sum(\overleftarrow{C}_n\times P_{b,c})\right)\end{aligned}$$

where the cycle $\overleftarrow{C}_n$ is obtained from the cycle $\overrightarrow{C}_n$ by reversing the orientation of the edges in $\overrightarrow{C}_n$. Hence

$$f_{(a,b)}\left(\sum(\overrightarrow{C}_n\times P_{a,b})\right)-f_{(c,b)}\left(\sum(\overrightarrow{C}_n\times P_{b,c})\right)=\sum_{i\in n} f[D_i]=0. \qquad \blacksquare$$

Let A and B be two graphs, $\overrightarrow{C}_n$ a cycle of A and (a,b,c,d) a walk in B. Assume that f is a homomorphism of $A\times B$ to some C_4-free graph G. Then using (2) and Lemma 2

$$\begin{aligned}&f_{(a,b)}\left(\sum(\overrightarrow{C}_n\times P_{a,b})\right)=\\ &\quad= f_{(c,b)}\left(\sum(\overrightarrow{C}_n\times P_{b,c})\right)=\sigma\left(f_{(b,c)}\left(\sum(\overrightarrow{C}_n\times P_{b,c})\right)\right)=\\ &\quad=\sigma\left(f_{(d,c)}\left(\sum(\overrightarrow{C}_n\times P_{c,d})\right)\right)=f_{(c,d)}\left(\sum(\overrightarrow{C}_n\times P_{c,d})\right).\end{aligned}$$

We note: If A, B, $\overrightarrow{C}_n$, f and a,b,c,d are as above then

$$f_{(a,b)}\sum(\overrightarrow{C}_n\times P_{a,b})=f_{(c,d)}\sum(\overrightarrow{C}_n\times P_{c,d}). \tag{4}$$

Lemma 3. *Let A and B be two graphs, with B connected and of chromatic number at least three; let f be a homomorphism of $A\times B$ into a C_4-free graph G; and let $\overrightarrow{C}_n$ be a cycle of A. Then*

$$f_{(a,b)}\sum(\overrightarrow{C}_n\times P_{a,b})=\sigma\left(f_{(a,b)}\sum(\overrightarrow{C}_n\times P_{a,b})\right)=f_{(c,d)}\sum(\overrightarrow{C}_n\times P_{c,d})$$

for any two edges $\{a,b\}$ and $\{c,d\}$ of B.

Proof. Let $(a_0,a_1,a_2,\ldots,a_{2l},a_{2l+1})$ with $a_0=a_{2l+1}=a$ be a "closed" walk of odd length in the graph B. Then according to equation (4), Lemma 2 and equation (2)

$$\begin{aligned} f_{(a_0,a_1)} \sum(\overrightarrow{C}_n \times P_{a_0,a_1}) &= f_{(a_2,a_3)} \sum(\overrightarrow{C}_n \times P_{a_2,a_3}) = \\ &= f_{(a_4,a_5)} \sum(\overrightarrow{C}_n \times P_{a_2,a_5}) = \ldots = f_{(a_{2l},a_0)} \sum(\overrightarrow{C}_n \times P_{a_{2l},a_0}) = \\ &= f_{(a_1,a_0)} \sum(\overrightarrow{C}_n \times P_{a_1,a_0}) = \sigma\left(f_{(a_0,a_1)} \sum(\overrightarrow{C}_n \times P_{a_0,a_1})\right). \end{aligned} \tag{5}$$

Because every edge of the graph B lies on a closed walk of odd length and B is connected, we obtain from (5)

$$f_{(c,d)} \sum(\overrightarrow{C}_n \times P_{a,b}) = f_{(a,b)} \sum(\overrightarrow{C}_n \times P_{a,b}) = \sigma(f_{(a,b)} \sum(\overrightarrow{C}_n \times P_{a,b}))$$

for any two edges $\{a,b\}$ and $\{c,d\}$ of B. ∎

Hence, Under the conditions of Lemma 3, the chain $f_{(a,b)} \sum(\overrightarrow{C}_n \times P_{a,b})$ is independent of the particular edge $\{a,b\}$ of B, as well as of the ordering of the vertices a and b. Thus *a fortiori*, $f\left(\sum(\overrightarrow{C}_n \times P_{a,b})\right)$ (the image of the latter chain under π) is independent of the particular edge $\{a,b\}$, as well as of the ordering of the vertices a and b.

Lemma 4. *Let A and B be two graphs, with B connected and of chromatic number at least three; let f be a homomorphism of $A\times B$ into a C_4-free graph G; and let $\overrightarrow{C}_n$ be a cycle of A. Then there exists a chain $\gamma^A(\overrightarrow{C}_n, f)$ so that, for any edge $\{a,b\}$ of B,*

$$2 \cdot \gamma^A(\overrightarrow{C}_n, f) = f\left(\sum(\overrightarrow{C}_n \times P_{a,b})\right)$$

Proof. We get from Lemma 3 that

$$f_{(a,b)} \sum(\overrightarrow{C}_n \times P_{a,b}) = \sigma(f_{(a,b)} \sum(\overrightarrow{C}_n \times P_{a,b})).$$

This implies that there is a matching of the terms of the chain $f_{(a,b)} \sum(\overrightarrow{C}_n \times P_{a,b})$ written in normal form so that every term of the form $((f((c_i,a)),0),(f((c_{i+1},b)),1))$ is matched with a term of the form $((f((c_j,a)),1),(f((c_{j+1},b)),0))$, where $f((c_i,a)) = f((c_j,a))$ and $f((c_{i+1},b)) = f((c_{j+1},b))$.

Let $\mathbf{u}$ be the sum of all terms of the form $((f((c_i,a)),0),(f((c_{i+1},b)),1))$ and $\mathbf{w}$ the sum of all terms of the form $((f((c_j,a)),1),(f((c_{j+1},b)),0))$. Then $\pi(\mathbf{v}) = \pi(\mathbf{w})$ and $\pi(\mathbf{v}) + \pi(\mathbf{w}) = f\left(\sum(\overrightarrow{C}_n \times P_{a,b})\right)$. Put $\gamma^A(\overrightarrow{C}_n, f) := \pi(\mathbf{v})$. ∎

It follows that, under the condition of Lemmas 3 and 4, $\gamma^A(\overrightarrow{C}_n, f)$ is independent of the particular edge $\{a,b\}$ of B as well as of the ordering of the vertices a and b. And in case A is also connected and of chromatic number at least three, if $\overrightarrow{C}_m$ is a cycle of B, then we similarly obtain the chain $\gamma^B(\overrightarrow{C}_m, f)$ with $2\cdot\gamma^B(\overrightarrow{C}_m, f) = f_{(a,b)} \sum(P_{a,b} \times \overrightarrow{C}_m)$ for any edge $\{a, b\}$ of A.

Lemma 5. *Let A and B be connected graphs with chromatic number at least three and f a homomorphism of $A \times B$ into a C_4-free graph G. If $\overrightarrow{C}_n$ is a cycle of A and $\overrightarrow{C}_m$ is a cycle of B with n and m odd, then the parity of $\|\gamma^A(\overrightarrow{C}_n, f)\|$ is different from the parity of $\|\gamma^B(\overrightarrow{C}_m, f)\|$.*

Proof. Let

$$\begin{aligned}
P^+ &:= \{\{(i,j),(i+1,j+1)\} \mid i \in n \,\text{and}\, j \in m\}\,,\\
\overrightarrow{P}^+ &:= \{((i,j),(i+1,j+1)) \mid i \in n \,\text{and}\, j \in m\}\,,\\
P^- &:= \{\{(i,j),(i+1,j-1)\} \mid i \in n \,\text{and}\, j \in m\}\,,\\
\overrightarrow{P}^- &:= \{((i,j),(i+1,j-1)) \mid i \in n \,\text{and}\, j \in m\}\,,
\end{aligned}$$

so that $P^+ \cup P^-$ be the set of edges of the subgraph $C_n \times C_m$ of $A \times B$. Note that each edge of P^+ has the same orientation in $\overrightarrow{C}_n \times C_m$ and in $C_n \times \overrightarrow{C}_m$, while each edge of P^- is oriented differently in $\overrightarrow{C}_n \times C_m$ and $C_n \times \overrightarrow{C}_m$. Now, using Lemma 3 and Lemma 4,

$$m \cdot 2 \cdot \gamma^A(\overrightarrow{C}_n, f) = \sum_{(a,b)\in E(\overrightarrow{C}_m)} f\left(\sum \overrightarrow{C}_n \times P_{a,b}\right) = f\left(\sum \overrightarrow{P}^+\right) + f\left(\sum \overrightarrow{P}^-\right)$$

and

$$n \cdot 2 \cdot \gamma^B(\overrightarrow{C}_m, f) = \sum_{(a,b)\in E(\overrightarrow{C}_n)} f\left(\sum P_{a,b} \times \overrightarrow{C}_m\right) = f\left(\sum \overrightarrow{P}^+\right) - f\left(\sum \overrightarrow{P}^-\right).$$

Hence

$$m \cdot \gamma^A(\overrightarrow{C}_n, f) + n \cdot \gamma^B(\overrightarrow{C}_m, f) = f\left(\sum \overrightarrow{P}^+\right).$$

Thus, given that the function mapping $\mathbf{v}$ to $\overline{\|\mathbf{v}\|}$ is a group homomorphism, that m and n are odd, and using Lemma 1 for the third from last equality:

$$\begin{aligned}
&\overline{\|\gamma^A(\overrightarrow{C}_n, f)\|} + \overline{\|\gamma^B(\overrightarrow{C}_m, f)\|}\\
&\quad = \overline{m} \cdot \overline{\|\gamma^A(\overrightarrow{C}_n, f)\|} + \overline{n} \cdot \overline{\|\gamma^B(\overrightarrow{C}_m, f)\|} =\\
&\quad = \overline{\|m \cdot \gamma^A(\overrightarrow{C}_n, f) + n \cdot \gamma^B(\overrightarrow{C}_m, f)\|} =\\
&\quad = \overline{\|f\left(\sum \overrightarrow{P}^+\right)\|} = \overline{\|\sum \overrightarrow{P}^+\|} = \overline{m \cdot n} = \overline{1}.
\end{aligned}$$

∎

It follows that, under the conditions of Lemma 5, the parity of $\|\gamma^A(\overrightarrow{C}_n, f)\|$ is equal to the parity of $\|\gamma^A(\overrightarrow{C}_{n'}, f)\|$ for any two odd cycles $\overrightarrow{C}_n$ and $\overrightarrow{C}_{n'}$ of A. This common parity is the *cycle parity of A under the homomorphism f*. Lemma 5 says that if A and B are two connected graphs of chromatic number at least three and if f is a homomorphism of $A\times B$ into a C_4-free graph G, then the cycle parity of A under the homomorphism f is different from the cycle parity of B under the homomorphism f.

Let f be a homomorphism of the product of two graphs to some graph G. Say $f : A\times B \mapsto G$. Imagine the vertices of $A\times B$ as a matrix, the rows are pairs of constant first coordinate and the columns are pairs of constant second coordinate. Fix $b\in V(B)$ and consider the column of all pairs of the form (x,b) with $x\in V(A)$. The homomorphism f restricted to this column can be viewed as a function f_b of $V(A)$ to $V(G)$. Of course f_b need not be a homomorphism. In general there will be vertices $a\in V(A)$ for which there is a vertex $x\in V(A)$ such that $a\sim x$ in A but $f_b(a)\not\sim f_b(x)$ in G. We denote by $[A\overset{f}{\times}b]_1$ this set of vertices of A (they are "bad" for f_b), and we denote by $[A\overset{f}{\times}b]_0$ the set of all other vertices of A. Formally, for every $b\in V(B)$, let

$$[A\overset{f}{\times}b]_1 := \{a \in V(A) \mid \exists\, x \in V(A)\ (\{a,x\} \in E(A) \text{ and } \{f(a,b), f(x,b)\} \notin E(G))\}$$

and $[A\overset{f}{\times}b]_0 := V(A)\setminus[A\overset{f}{\times}b]_1$. So

$$[A\overset{f}{\times}b]_0 = \{a \in V(A) \mid \forall x \in V(A)\ \ (\{a,x\} \in E(A) \rightarrow \{f(a,x), f(b,x)\} \in G)\}\,.$$

The following Lemmas 6 and 7 are simple but important general observations about the set $[A\overset{f}{\times}b]_1$ already observed in [2].

Lemma 6. *Let f be a homomorphism of the graph $A\times B$ to the graph G and $\{b,c\}\in E(B)$. If $a\in[A\overset{f}{\times}b]_1$ then $f(a,b)\neq f(a,c)$.*

Proof. Assume for a contradiction that $f(a,b)=f(a,c)$. Because $a\in[A\overset{f}{\times}b]_1$ there is $x \in V(A)$ so that $\{a,x\} \in E(A)$ but $\{f(a,b), f(x,b)\} \notin E(G)$. It follows that

$$\{f(a,c), f(x,b)\} = \{f(a,b), f(x,b)\} \notin E(G).$$

This is a contradiction because f is a homomorphism of $A \times B$ to G and $\{(a,c),(x,b)\}\in E(A\times B)$. ∎

Lemma 7. *Let f be a homomorphism of the graph $A\times B$ to the graph G. If there is no homomorphism of A to G and B does not contain an isolated point then there exists for every vertex $b\in V(B)$ an odd cycle, say $\overrightarrow{C}^b$, so that*

$$V\left(\overrightarrow{C}^b\right) \subseteq [A\overset{f}{\times}b]_1.$$

Proof. Assume that $\{b,c\}\in E(B)$ and that there is no such odd cycle $\overrightarrow{C}^b$ in $[A\overset{f}{\times}b]_1$. We will construct a homomorphism g of A to G.

Because there is no odd cycle $\overrightarrow{C}^b$ there are two disjoint sets L and R with $L\cup R=[A\overset{f}{\times}b]_1$ and $([L]^2\cup[R]^2)\cap E(A)=\emptyset$. Let $g\colon V(A)\mapsto V(G)$ be given by

$$g(a) = \begin{cases} f(a,b), & \text{for } a \in [A\overset{f}{\times}b]_0 \cup L, \\ f(a,c), & \text{for } a \in R. \end{cases}$$

For $\{x,y\}\in E(A)$ we have to show that $\{g(x),g(y)\}\in E(G)$. If both x and y are elements of $[A\overset{f}{\times}b]_0\cup L$ then at least one of x and y, say x, is an element of $[A\overset{f}{\times}b]_0$. It follows from the definitions of g and $[A\overset{f}{\times}b]_0$ that

$$\{g(x),g(y)\} = \{f(x,b),f(y,b)\} \in E(G).$$

If $x\in[A\overset{f}{\times}b]_0\cup L$ and $y\in R$ then

$$\{g(x),g(y)\} = \{f(x,b),f(y,c)\} \in E(G)$$

because $\{(x,b),(y,c)\}\in E(A\times B)$ and f is a homomorphism. ∎

Theorem. *Let A and B be connected graphs, each containing a triangle, and let f a homomorphism of $A\times B$ to a C_4-free graph G. Then either $A\mapsto G$ or $B\mapsto G$.*

Proof. We may assume without loss that A has odd cycle parity under the homomorphism f and that $\{b,c,d\}$ is a triangle in B. If $A\not\mapsto G$ let $\overrightarrow{C}^b=\langle 0,1,2,\dots,n-1,0\rangle$ be an odd cycle of A with

$$V(\overrightarrow{C}^b) \subseteq [A\overset{f}{\times}b]_1$$

given by Lemma 7.

We obtain, for every $i\in n$ the “square” $\{(i,b)\sim(i+1,d)\sim(i,c)\sim(i-1,d)\}$ in $A\times B$. Because G is C_4-free and f a homomorphism it follows that either $f(i,b)=f(i,c)$ or $f(i+1,d)=f(i-1,d)$ and hence from Lemma 6 that

$f(i{+}1,d)=f(i{-}1,d)$. We conclude that f is constant on the set $\{(i,d)\,|\,i\in n\}$. Similarly we obtain that f is constant on the set $\{(i,c)\,|\,i\in n\}$. Hence

$$f\left(\sum\left(\overrightarrow{C}^b\times P_{c,d}\right)\right)=0 \text{ and therefore } \gamma^A(\overrightarrow{C}^b, f\ )=0.$$

But this would imply that A has even cycle parity under the homomorphism f. ∎

If the A_i's $i\in I$ are graphs with pairwise disjoint vertex sets and edge sets, then we denote by $\sum_I A_i$ the graph with vertex set $\bigcup_I V(A_i)$ and edge set $\bigcup_I E(A_i)$. The theorem stated in the introduction follows from the fact that $(\sum_I A_i)\times(\sum_J B_j)=\sum_{I\times J}(A_i\times B_j)$: Assume that for each $(i,j)\in I\times J$, $A_i\mapsto G$ or $B_j\mapsto G$; if $\sum_I A_i\not\mapsto G$, then for some $i\in I$, $A_i\not\mapsto G$, thus, for every $j\in J$, $B_j\mapsto G$, hence $\sum_J B_j\mapsto G$.

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Christian Delhommé

E.R.M.I.T.
Département de Mathématiques
et d'Informatique
Université de La Réunion
15, avenue René Cassin, BP 71551
97715 Saint-Denis Messag.
Cedex 9, FRANCE
delhomme@univ-reunion.fr

Norbert Sauer

University of Calgary
Department of Mathematics
and Statistics
2500 University Dr. N. W.
Calgary, Alberta,
T2N 1N4 Canada
nsauer@math.ucalgary.ca